THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. NOVEMBER, 1898. No. 11.

POINT INVARIANTS FOR THE FINITE CONTINUOUS GROUPS OF THE PLANE.

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In an article entitled "Theorie der Transformationsgruppen," Mathematische Annalen, Bd. XVI., Lie has classified and reduced to canonical forms all the *finite continuous* groups of the plane. We may very appropriately call these the *Lie groups*. In the present paper it is proposed to determine the functions of the coördinates of n points which remain invariant by these groups. In some cases the computations are quite complex; only the methods of calculation are given.

An infinitesimal point transformation gives to x and y the increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

respectively, where δt is independent of x and y. Such an infinitesimal transformation is denoted in the Lie notation by

$$Xf \equiv \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}. \tag{1}$$

The variation of any function $\varphi(x, y)$ by this transformation is given by $X\varphi(x, y)\delta t$. Hence, if $\varphi(x, y)$ is to remain invariant by the transformation Xf, its variation must be zero, or

$$Xq \equiv \xi(x, y)\frac{\partial q}{\partial x} + \eta(x, y)\frac{\partial q}{\partial y} = 0.$$

The function $\varphi(x, y)$, invariant by the infinitesimal transformation Xf, is determined as a solution of the linear partial differential equation $Xf=0$.

The infinitesimal transformation Xf may be *extended* to include the increments of any number of points, x_i, y_i, $i=1, 2 \ldots\ldots n$. We may write this extended transformation as

$$Wf \equiv \sum_1^n iX^{(i)}f \equiv \sum_1^n i\xi(x_i, y_i)\frac{\partial f}{\partial x_i} + \sum_1^n i\eta(x_i, y_i)\frac{\partial f}{\partial y_i}. \tag{2}$$

The functions of the coördinates of n points invariant by Wf will be the $2n-1$ independent solutions of $Wf=0$. n of these solutions may be selected in the form $\varphi(x_i, y_i)$, where $\varphi(x, y)$ is a solution of $Xf=0$; the remaining $n-1$ solutions will in general differ from $\varphi(x, y)$ in form.*

Infinitesimal transformations are called *independent* when no one can be expressed as a linear function of the others with constant coefficients. r such independent infinitesimal transformations

$$X_k f \equiv \xi_k(x, y)\frac{\partial f}{\partial x} + \eta_k(x, y)\frac{\partial f}{\partial y}, \quad (k=1 \ \ldots\ldots r),$$

form a group when

$$X_i(X_k f) - X_k(X_i f) \equiv \sum_1^r s c_{iks} X_s f, \quad (c_{iks}=\text{constants}). \tag{3}$$

The transformations of this *r-parameter* group $X_k f$ may be extended after the manner of (2) above, giving us $W_k f$ which determine the increments of a function $f(x_1, y_1; x_2, y_2; \ldots\ldots x_n, y_n)$. On account of relation (3),

$$W_1 f=0, \ W_2 f=0, \ \ldots\ldots W_r f=0$$

are known to form a *complete* system of linear partial differential equations in $2n$ variables $x_1 y_1, x_2 y_2, \ldots\ldots x_n y_n$, with at least $2n-r$ independent solutions. If the r-rowed determinants of the coefficients vanish, more than $2n-r$ independent solutions will exist. These solutions are the invariants of the coördinates of n points by the r-parameter group of $X_k f$.

According to the theory here outlined we shall determine the *point-invariants* of the finite continuous groups as classified by Lie in the Mathematische Annalen, Vol. XVI.

*Lie: Theorie der Transformationsgruppen, Bd. I., §59.

§ 1.
Invariants of the Primitive Groups.

The *primitive* groups of the plane leave *no* family of ∞^1 curves invariant, and may be reduced by a proper choice of variables to some one of the canonical forms known as (1) special linear, (2) general linear, (3) general projective.

1. *The special linear group*

$$\boxed{p,\ q,\ xq,\ xp-yq,\ yp}.$$

Here $p=\frac{\partial f}{\partial x}$, $q=\frac{\partial f}{\partial y}$. The increments of $f(x_i, y_i)$ by the members of this group *extended* are given by

$$W_1 f\equiv\sum_1^n i\frac{\partial f}{\partial x_i},\quad W_2 f\equiv\sum_1^n i\frac{\partial f}{\partial y_i},\quad W_3 f\equiv\sum_1^n ix_i\frac{\partial f}{\partial y_i},$$

$$W_4 f\equiv\sum_1^n i(x_i\frac{\partial f}{\partial x_i}-y_i\frac{\partial f}{\partial y_i}),\quad W_5 f\equiv\sum_1^n iy_i\frac{\partial f}{\partial x_i}.$$

The invariant functions sought will be the $2n-5$ independent solutions of the complete system

$$W_1 f=0,\ W_2 f=0,\ W_3 f=0,\ W_4 f=0,\ W_5 f=0. \qquad (s)$$

The first two equations show the solutions to be functions of

$$\varphi_j=x_1-x_j,\ \psi_j=y_1-y_j,\ (j=2\ldots\ldots n).$$

The remaining equations then take the form

$$\sum_2^n j\varphi_j\frac{\partial f}{\partial f_j}=0,\quad \sum_2^n j\,(\varphi_j\frac{\partial f}{\partial\varphi_j}-\psi_j\frac{\partial f}{\partial\psi_j})=0,\quad \sum_2^n j\,\psi_j\frac{\partial f}{\partial\varphi_j}=0.$$

The second of these equations has solutions

$$\Phi_j=\varphi_j\psi_j,\quad \Psi_k=\varphi_2\psi_k,\quad (k=3\ldots\ldots n).$$

With these functions as new variables, the first and third equations become

$$\frac{\partial f}{\partial\phi_2}+\sum_3^n k\,\frac{\phi_k}{\psi_k}\left\{\frac{\phi_k}{\psi_k}\frac{\partial f}{\partial\phi_k}+\frac{\partial f}{\partial\psi_k}\right\}=0, \qquad (1)$$

$$\frac{\phi_2}{\psi_2}\frac{\partial f}{\partial\phi_2}+\frac{1}{\phi_2\psi_2}\sum_3^n k\psi_k{}^2\frac{\partial f}{\partial\varphi_k}+\frac{1}{\psi_2}\sum_3^n k\psi_k\frac{\partial f}{\partial\psi_k}=0. \qquad (2)$$

The solutions of (1) are seen to be

Note: φ, ψ in (1), (2) should be Φ, Ψ.

$$\sigma_k=\frac{\psi_k}{\varphi_k},\ \rho_k=\varphi_2-\frac{\psi_k^2}{\varphi_k},$$

while (2) takes the form

$$\sum_3^n k\left(\sigma_k\rho_k\frac{\partial f}{\partial\sigma_k}+\rho_k{}^2\frac{\partial f}{\partial\rho_k}\right)=0,$$

with solutions

$$\Delta_k=-\frac{\rho_k}{\sigma_k},\ I_e=\frac{1}{\rho_4}-\frac{1}{\rho_3},\ (k=3\ldots\ldots n,\ l=4\ldots\ldots n).$$

Since any functions of Δ and I will be solutions of our complete system (s), we may choose

$$\Delta_k=-\frac{\rho_k}{\rho_2}=|\,1\ 2\ k\,|,\ D_l=I_l.\Delta_3.\Delta_l=|\,1\ 3\ l\,|,\ \text{where}\ |\,i\ j\ k\,|=\begin{vmatrix} x_i & y_i & 1\\ x_j & y_j & 1\\ x_k & y_k & 1\end{vmatrix},$$

as solutions of (s), and, therefore, as the $2n-5$ invariants of the coördinates of n points.

The forms of Δ and D show that *the special linear group leaves invariant all areas.*

2. *The general linear group*

$$\boxed{p,\ q,\ xq,\ xp-yq,\ yp,\ xp+yq}$$

This group furnishes a complete system of six linear partial differential equations, the first five equations of the system being identical with those of the preceding section. We need only to determine the functions of Δ_k and D_l which satisfy

$$\sum_1^n i\ (x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=0.$$

This equation requires that x, y enter in the final solutions with the degree zero. Hence, we may write at once the invariants :

$$I_l=\frac{\Delta_l}{\Delta_3}=|\,1\ 2\ l\,|:|\,1\ 2\ 3\,|$$

$$J_l=\frac{D_l}{\Delta_3}=|\,1\ 3\ l\,|:|\,1\ 2\ 3\,|\quad(l=4,\ \ldots\ n).$$

Hence, *by the general linear group the ratio of areas remain constant.*

3. *The general projective group*

$$\boxed{p,\ q,\ xq,\ xp-yq,\ yp,\ xp+yq,\ x^2p+xyq,\ xyp+y^2q}\ .$$

The members of this group extended and equated to zero give a complete system of eight partial differential equations, the first six of which are identical with those of the general linear group, and therefore have solutions I, J defined above. The last two equations,

$$\sum_1^n i x_i\left(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i}\right)=\sum_1^n i y_i\left(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i}\right)=0,$$

expressed in terms of I and J, become somewhat complex. For brevity we shall write these symbolically as

$$Uf=0,\quad Vf=0,$$

where $Uf=0$ has the solutions

$$I_4,\quad \Phi_m=\frac{I_m J_4}{J_m},\quad \Psi_m=\frac{\phi_m+I_m(I_4-\phi_m-1)}{I_m(J_4-I_4+1)},\quad (m=5.....n).$$

With I_4, ϕ_m and ψ_m as new variables, the equation $Vf=0$ takes the simple form

$$I_4\frac{\partial f}{\partial I_4}+\sum_5^n m\varphi_m\frac{\partial f}{\partial \varphi_m}=0,$$

whose solutions are clearly

$$\theta_m=\frac{\varphi_m}{I_4},\quad \Psi_m,\quad (m=5.....n).$$

Selecting as invariants θ_m and $H_m=\dfrac{1+\Psi_m}{\theta_m}$, and restoring the variables x_i, y_i we have

$$\theta_m=\frac{|\,1\;2\;m\,|\,.\,|\,1\;3\;4\,|}{|\,1\;2\;4\,|\,.\,|\,1\;3\;m\,|},\quad H_m=\frac{|\,1\;2\;4\,|\,.\,|\,2\;3\;m\,|}{|\,1\;2\;m\,|\,.\,|\,2\;3\;4\,|}.$$

The forms of θ and H show *that the general projective group leaves invariant the cross-ratios of five points.* Five points have two independent invariants, θ, H; four points have no invariant unless they be collinear, in which case the invariant is the cross-ratio of the four points.

§ 2.

Invariants of such Imprimitive Groups as Leave Unchanged One Family of ∞^1 Curves.

The remaining finite continuous groups of the plane are known as *imprimitive*, and are classified according as they leave invariant *one*, *two*, or an *infinite number* of families of ∞^1 curves.* The groups of the first category have been

*Lie: Math. Annalen, Bd. XVI.

reduced by Lie to such canonical forms that the family of curves x=constant is transformed into itself.

4. $$\boxed{\begin{array}{c} X_1 q,\ X_2 q,\ X_3 q,\ \ldots\ldots X_r q \\ r>1 \end{array}}.$$

Here X_k is a function of x alone. This group leaves the curves of the family $x=a$ singly invariant.

The complete system

$$W_k f \equiv \sum_1^n i X_k(x_i) \frac{\partial f}{\partial y_i} = 0, \quad (k=1 \ \ldots\ldots \ r),$$

corresponding to this group, has as solutions $x_1, x_2, \ldots\ldots x_n$ and $n-r$ other independent functions Δ_s, $(s=1\ldots\ldots\ n-r)$, which we shall define as the $n-r$ determinants of the matrix

$$\left\| \begin{array}{llll} y_1 & y_2 & y_3 & \ldots\ldots y_{r+s} \ \ldots\ldots y_n \\ X_1(x_1) & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ X_2(x_1) & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ X_r(x_1) & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{array} \right\|, \qquad (M_r)$$

$$(n \geqq r+1)$$

formed by filling the $(r+1)$th column successively by the $(r+1)$, the $(r+2)\ldots$, and the nth.

Our invariants are x_i and Δ_s, $(i=1\ldots\ldots n,\ s=1\ldots\ldots n-r)$.

5. $$\boxed{\begin{array}{c} X_1 q,\ X_2 q \ldots\ldots X_{r-1} q,\ yq \\ r>2 \end{array}}.$$

This group furnishes the complete system

$$W_k f \equiv \sum_1^n i X_k(x_i) \frac{\partial f}{\partial y_i} = 0, \quad Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0, \quad (k=1\ldots\ldots r-1).$$

The solutions of $W_k f=0$ are clearly $x_1, x_2, \ldots\ldots x_n$ and the determinants Δ_s, $(s=0, 1, \ldots\ldots n-r)$, of the matrix (M_{r-1}) defined above. $Yf=0$ requires the ratios of y_i to appear. We may then write as invariants of the group

$$x_1, x_2, \ldots\ldots x_n \text{ and } \rho_t = \frac{\Delta_t}{\Delta_o}, \quad (t=1\ldots\ldots n-r).$$

6.

$$\varepsilon^{\alpha_k x} q,\ x\varepsilon^{\alpha_k x} q,\ x^2\varepsilon^{\alpha_k x} q,\ \ldots\ldots x^{\rho_k}\varepsilon^{\alpha_k x} q,\ p$$

$$k=1\ldots\ldots m,\ \sum_1^m k\rho_k + m = r-1,\ r>2$$

We have

$$W_k{}^{t_k} f \equiv \sum_1^n i(x_i)^{t_k} \cdot \varepsilon^{\alpha_k x_i} \frac{\partial f}{\partial y_i} = 0,\ (k=1,\ \ldots\ldots m,\ t_k=0,\ 1\ldots\ldots \rho_k).$$

$$Xf \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The last equation requires the functions

$$\varphi_j = x_j - x_1 \quad (j=2\ldots\ldots n),$$

to appear in the solutions of the system. On dividing the remaining equations, respectively, by $\varepsilon^{\alpha_k x_1}$, the exponents of ε all become functions of φ_j. The independent determinants Δ_s, $(s=0,\ 1\ldots\ldots n-r)$ of the matrix (M_{r-1}), formed as in 4, will be solutions.

The invariants are, therefore, φ_j and Δ_s.

7.

$$\varepsilon^{\alpha_k x} q,\ x\varepsilon^{\alpha_k x} q,\ x^2\varepsilon^{\alpha_k x} q,\ \ldots\ldots\ x^{\rho_k}\varepsilon^{\alpha_k x} q,\ yq,\ p$$

$$k=1,\ \ldots\ldots\ m,\ \sum_1^n k\rho_\kappa + m = r-2,\ r>3$$

The complete system given by this group is

$$W_k{}^{t_k} f \equiv \sum_1^n i(x_i)^{t_k} \cdot \varepsilon^{\alpha_k x_i} \frac{\partial f}{\partial y_i} = 0,\ (k=1\ldots\ldots m,\ t_k=0,\ 1,\ \ldots\ldots \rho_k).$$

$$Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0,\ Xf \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

If a matrix be constructed, as indicated in 4 and 6, from the coefficients of the first $r-2$ equations, it will be observed that the independent determinants Δ_s, $(s=-1,\ 0,\ 1\ldots\ldots n-r)$, will be linear and homogeneous in y_i with coefficients composed of functions of $\varphi_j = x_j - x_1$. Δ_s will be solutions of all equations except $Yf=0$, which requires the ratios of y_i to appear. Hence, the invariants may be written

$$\varphi_j = x_j - x_1,\ \rho_t = \frac{\Delta_t}{\Delta_{-1}},\ (j=2\ldots\ldots n,\ t=0\ldots\ldots n-r).$$

8. $$\boxed{\begin{array}{c} q,\quad xq,\quad x^2q,\ \ldots\ldots x^{r-3}q,\quad p,\quad xp+cyq \\ r>3 \end{array}}.$$

Here the complete system is

$$W_k f \equiv \sum_1^n i x_i^k \frac{\partial f}{\partial y_i} = 0,\ (k=0,\ 1\ \ldots\ldots r-3)$$

$$Yf \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} + c \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0,\ Xf \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The solutions

$$\psi_j = y_1 - y_j,\ \varphi_j = x_1 - x_j,\ (j=2\ldots\ldots n),$$

of the first and last of these equations, introduced in Yf, give

$$Y'f \equiv \sum_2^n j \left(\varphi_j \frac{\partial f}{\partial \varphi_j} + c\,\psi_j \frac{\partial f}{\partial \psi_j}\right) = 0,$$

whose solutions are

$$\sigma_k = \frac{\varphi_k}{\varphi_2},\quad \rho_j = \frac{\psi_j}{(\varphi_j)^c},\quad (k=3\ldots\ldots n,\ j=2\ldots\ldots n).$$

$W_k f$, expressed in σ, ρ, is

$$W_t' f \equiv \frac{\partial f}{\partial \rho_2} + \sum_3^n k \sigma_k^{t-c} \frac{\partial f}{\partial \rho_k} = 0,\ (t=1,\ \ldots\ldots r-3).$$

The solutions of these $r-3$ equations are σ_k, and the determinants Δ_s, $(s=1, 2\ldots\ldots n-r+2)$, of the matrix

$$\left\| \begin{array}{cccc} \rho_2 & \rho_3 & \rho_4 \ldots\ldots \rho_{r-3+s} \ldots\ldots & \rho_n \\ 1 & \sigma_3^{1-c} & \sigma_4^{1-c} \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots & & & \\ 1 & \sigma_3^{r-3-c} & \sigma_4^{r-3-c} \ldots\ldots\ldots\ldots & \sigma_n^{r-3-c} \end{array} \right\|.$$

Our invariants are

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2} \text{ and } \Delta_s.$$

9. $$\boxed{\begin{array}{c} q,\quad xq,\quad x^2q,\ \ldots\ldots\ x^{r-3}q,\quad p,\quad xp+[(r-2)y+x^{r-2}]q \\ r>2 \end{array}}.$$

The solutions of the complex system,

$$W_k f \equiv \sum_1^n i x_i^k \frac{\partial f}{\partial y_i} = 0, \quad (k=0, 1 \ldots\ldots r-3),$$

$$Yf \equiv \sum_1^n i\{x_i \frac{\partial f}{\partial x_i} + [(r-2)y_i + x_i^{r-2}] \frac{\partial f}{\partial y_i}\} = 0,$$

$$Xf \equiv \sum_1^n i \frac{\partial f}{dx_i} = 0,$$

may be obtained in a manner similar to 8. The solutions, ψ_j, φ_j, of the first and last equations, introduced in Yf as variables, give

$$Y'f \equiv \sum_2^n j\{\varphi_j \frac{\partial f}{\partial \varphi_j} + [(r-2)\psi_j + \varphi_j^{r-2}] \frac{\partial f}{\partial \psi_j}\} = 0,$$

with solutions,

$$\sigma_k = \frac{\varphi_k}{\varphi_2}, \quad \rho_j = \log \varphi_j - \frac{\psi_j}{\varphi_j^{r-2}}, \quad (k=3 \ldots\ldots n, j=2 \ldots\ldots n).$$

Introducing σ, ρ as new variables in Wf, we have

$$W_t' f \equiv \frac{\partial f}{\partial \rho_2} + \sum_3^n k \sigma_k^{t+2-r} \frac{\partial f}{\partial \rho_k} = 0,$$

whose solutions are σ_k and Δ_s of the matrix constructed as in 8.

The invariants are, therefore,

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad \Delta_s, \quad (k=3 \ldots\ldots n, \ s=1 \ldots\ldots n-r+2).$$

10. $\boxed{\begin{array}{c} q, \ xq, \ x^2 q, \ \ldots\ldots x^{r-4} q, \ yq, \ p, \ xp \\ r>3 \end{array}}$.

For this group

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0, \ 1 \ldots\ldots r-4),$$

$$Yf \equiv \sum_1^n i y_i \frac{\partial f}{\partial y_i} = 0, \quad X_1 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0, \quad X_2 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0.$$

The last two equations show that the ratios of the differences of the x's, say

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad (k=3, \ldots\ldots n),$$

shall appear in the final solutions. The $n-r+3$ independent determinants Δ_s, $(s=0, 1 \ldots\ldots n-r+2)$, of the matrix

$$\left\| \begin{array}{llll} y_1 & y_2 & y_3 \ldots\ldots & y_{r-2+s} \ldots\ldots y_n \\ 1 & 1 & 1 \ldots\ldots 1 & \ldots\ldots 1 \\ x_1 & x_2 & x_3 \ldots\ldots & \ldots\ldots x_n \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ x_1^{r-4} & x_2^{r-4} \ldots\ldots & \ldots\ldots x_{r-2+s}^{r-4} & \ldots\ldots x_n^{r-4} \end{array} \right\|$$

are solutions of the first $r-3$ equations. These determinants are at the same time homogeneous in y_i and $x_i - x_k$; their ratios will, therefore, satisfy the requirements of

$$\sigma_k = \frac{x_1 - x_k}{x_1 - x_2} \quad \text{and} \quad Yf = 0.$$

Hence, we may write our $2n-r$ invariants as

$$\sigma_k \text{ and } \rho_t = \frac{\Delta_t}{\Delta_0}, \quad (k=3 \ldots\ldots n, \; t=1 \quad \ldots\ldots n-r+2).$$

11. $q, \quad xq, \quad x^2q, \ldots\ldots x^{r-4}q, \quad p, \quad 2xp+(r-4)yq, \quad x^2p+(r-4)xyq$ $\quad r>4$.

From this group we obtain the differential equations

$$W_t f \equiv \sum_1^n i x_i^t \frac{\partial f}{\partial y_i} = 0, \quad (t=0, \; 1 \ldots\ldots r-4),$$

$$X_1 f \equiv \sum_1^n i (2x_i \frac{\partial f}{\partial x_i} + (r-4)\, y_i \frac{\partial f}{\partial y_i}) = 0,$$

$$X_2 f \equiv \sum_1^n i \; (x_i^2 \frac{\partial f}{\partial x_i} + (r-4)\, x_i y_i \frac{\partial f}{\partial y_i}) = 0,$$

$$X_0 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

With the solutions ψ_j, φ_j, of the first and last of these equations, $X_2 f$ takes the form

$$X_2' f \equiv \sum_2^n j (\varphi_j^2 \frac{\partial f}{\partial \varphi_j} + (r-4)\, \varphi_j \psi_j \frac{\partial f}{\partial \psi_j}) = 0,$$

whose solutions may be selected,

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k},\ v_j=\frac{\psi_j}{\varphi_j{}^{r-4}},\ (k=3\ldots\ldots n,\ j=2\ldots\ldots n).$$

$X_1 f$ then becomes

$$X_1{}' f\equiv 2\sum_3^n k u_k\frac{\partial f}{\partial u_k}+(r-4)\sum_2^n j v_j\frac{\partial f}{\partial v_j}=0,$$

with solutions

$$\sigma_l=\frac{u_l}{u_3},\quad \rho_k=\frac{v_k}{u_k{}^{\frac{1}{3}(r-4)}},\quad \rho_2=\frac{v_2}{u_3{}^{\frac{1}{3}(r-4)}},\quad (l=4\ \ldots\ldots n,\ k=3\ \ldots\ldots n).$$

The remaining equations Wf in terms of σ, ρ are

$$\frac{\partial f}{\partial\rho_2}+\frac{\partial f}{\partial\rho_3}+\sum_4^n l(\sigma_l)^{-\alpha}\frac{\partial f}{\partial\rho_l}=0,\ [\alpha=\tfrac{1}{3}(r-4)],$$

$$W_t{}' f\equiv\frac{\partial f}{\partial\rho_3}+\sum_4^n l(\sigma_l)^{t-\alpha}\frac{\partial f}{\partial\rho_l}=0,\ (t=1,\ \ldots\ldots r-5).$$

Constructing a matrix,

$$\left\|\begin{array}{lll} \rho_2 & \rho_3 & \rho_4\ \ldots\ldots\rho_{r-3+s}\ldots\ldots\rho_n \\ 1 & 1 & \sigma_4{}^{-\alpha}\ldots\ldots\ldots\ldots\ldots \\ 0 & 1 & \sigma_4{}^{1-\alpha}\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ 0 & 1 & \sigma_4{}^{r-5-\alpha}\ldots\ldots\ldots\ldots\ldots \end{array}\right\|$$

our invariants may be written

$$\sigma_l=\frac{x_2-x_l}{x_1-x_l}:\frac{x_2-x_3}{x_1-x_3},\ \text{and}\ \Delta_s,\ (l=4\ldots\ldots n,\ s=1\ldots\ldots n-r+3).$$

12.

$q,\quad xq,\quad x^2q,\ \ldots\ldots x^{r-5}q,\quad yq,\quad p,\quad xp,\quad x^2p+(r-5)xyq$
$r>5$

This group furnishes the system

$$W_t f\equiv\sum_1^n i x_i{}^t\frac{\partial f}{\partial y_i}=0,\ (t=0,\ 1\ldots\ldots r-5),\ Yf\equiv\sum_1^n i y_i\frac{\partial f}{\partial y_i}=0,$$

$$X_2 f \equiv \sum_1^n i\,(x_i^2 \frac{\partial f}{\partial x_i} + (r-5)\, x_i y_i \frac{\partial f}{\partial y_i}) = 0,$$

$$X_1 f \equiv \sum_1^n i x_i \frac{\partial f}{\partial x_i} = 0, \qquad X_0 f \equiv \sum_1^n i \frac{\partial f}{\partial x_i} = 0.$$

The first and last two equations show the final solutions to be functions of

$$u_k = \frac{x_1 - x_k}{x_1 - x_2}, \quad \psi_j = y_1 - y_j, \quad (k = 3 \ldots\ldots n,\ j = 2. \ \ldots . n).$$

$X_2 f$ now becomes

$$X_2' f \equiv \sum_3^n k u_k (u_k - 1) \frac{\partial f}{du_k} + (r-5) \sum_3^n k u_k \frac{\partial f}{\partial \psi_k} + (r-5) \psi_2 \frac{\partial f}{\partial \psi_2} = 0,$$

with solutions

$$\sigma_l = \frac{u_3 (u_l - 1)}{u_l (u_3 - 1)}, \quad \rho_k = \frac{\psi_k}{(u_k - 1)^{r-5}}, \quad \rho_2 = \psi_2 \left(\frac{u_3}{u_3 - 1}\right)^{r-5}, \quad (l = 4 \ \ldots\ldots n,\ k = 3 \ \ldots\ldots n).$$

The remaining equations expressed in the new variables take the form

$$\left.\begin{aligned} &\frac{\partial f}{\partial \rho_2} + \frac{\partial f}{\partial \rho_3} + \sum_4^n l\,(\sigma_l)^{5-r} \frac{\partial f}{\partial \rho_l} = 0, \\ &W_t' f \equiv \frac{\partial f}{\partial \rho_3} + \sum_4^n l (\sigma_l)^{-t} \frac{\partial f}{\partial \rho_l} = 0, \quad (t = 1, \ \ldots\ldots r-4). \end{aligned}\right\} (A).$$

$$Y' f \equiv \sum_2^n j \rho_j \frac{\partial f}{\partial \rho_j} = 0.$$

The independent determinants Δ_s, $(s = 0, 1 \ldots . \ n-r+3)$, of the matrix formed from equations (A) as in 11 will be solutions of (A). Δ_s will be linear in ρ, but $Y' f$ requires the ratios of ρ's to enter in the solutions. We may, therefore, write our invariants

$$\sigma_l = \frac{x_2 - x_l}{x_1 - x_l} \div \frac{x_2 - x_3}{x_1 - x_3},$$

the cross-ratios of the abscissas of four points, and

$$\theta_t = \frac{\Delta_t}{\Delta_0}, \quad (t = 1 \ \ldots\ldots n - r + 3).$$

13. $\boxed{p, \ 2xp + yq, \ x^2 p + xyq}$

From this projective group we may obtain the complete system

$$\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n i\,(2x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i})=\sum_1^n i\,(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

The solutions y_i and $\varphi_j=x_1-x_j$ of the first of these equations introduced in the last give

$$\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial\varphi_j}+y_j\varphi_j\frac{\partial f}{\partial y_j})=0,$$

whose solutions are clearly

$$u_k=\frac{1}{\varphi_2}-\frac{1}{\varphi_k},\quad v_j=\frac{\varphi_j}{y_j},\quad y_1.$$

The second equation is now

$$y_1\frac{\partial f}{\partial y_1}-2\sum_3^n ku_k\frac{\partial f}{\partial u_k}+\sum_2^n jv_j\frac{\partial f}{\partial v_j}=0,$$

with solutions

$$\sigma_l=\frac{u_l}{u_3},\quad \rho_k=\frac{v_k}{v_2},\quad \rho_2=\frac{v_2}{y_1},\quad \rho_1=y_1u_3v_2,\quad (k=3\ldots\ldots n,\ l=4\ldots\ldots n).$$

Our invariants are then

$$\sigma_l=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_3}{x_1-x_l},\quad \rho_k=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\quad \rho_2=\frac{x_1-x_2}{y_1-y_2},\quad \rho_1=\frac{x_3-x_2}{x_3-x_1}:\frac{y_2}{y_1},$$

whose geometric significance is apparent.

14. $\quad yq,\quad p,\quad xp,\quad x^2p+xyq$

This four-parameter projective group yields the complete system

$$\sum_1^n iy_i\frac{\partial f}{\partial y_i}=\sum_1^n i\frac{\partial f}{\partial x_i}=\sum_1^n ix_i\frac{\partial f}{\partial x_i}=\sum_1^n i(x_i^2\frac{\partial f}{\partial x_i}+x_iy_i\frac{\partial f}{\partial y_i})=0.$$

Here, we introduce the solutions,

$$\varphi_j=x_1-x_j,\quad \psi_j=\frac{y_j}{y_1},$$

of the first two equations in the last two as new variables, and have

$$\sum_1^n j\,\varphi_j\frac{\partial f}{\partial\varphi_j}=\sum_2^n j(\varphi_j^2\frac{\partial f}{\partial\varphi_j}+\varphi_j\psi_j\frac{\partial f}{\partial\psi_j})=0.$$

The last of these new equations is satisfied by

$$\sigma_k=\frac{1}{\varphi_1}-\frac{1}{\varphi_k},\quad \rho_j=\frac{\varphi_j}{\psi_j},$$

while the first becomes

$$\sum_3^n k\sigma_k\frac{\partial f}{\partial\sigma_k}-\sum_2^n j\rho_j\frac{\partial f}{\partial\rho_j}=0.$$

The invariants of this group are, therefore,

$$u_l=\frac{\sigma_l}{\sigma_3}=\frac{x_2-x_l}{x_2-x_3}:\frac{x_1-x_l}{x_1-x_3},\ (l=4\ldots\ldots n),$$

$$v_k=\frac{\rho_k}{\rho_2}=\frac{x_1-x_k}{x_1-x_2}:\frac{y_k}{y_2},\ (k=3\ldots\ldots n),$$

$$v_2=\sigma_3\,\rho_2=\frac{x_2-x_3}{x_1-x_3}:\frac{y_2}{y_1}.$$

§ 3.

Invariants of Such Imprimitive Groups as Leave Unchanged More Than One Family of ∞^1 Curves.

The groups of this section are classified according as they leave invariant:

(*A*) Two families of ∞^1 curves: $x=$constant, $y=$constant.

(*B*) A single infinity of families of ∞^1 curves: $ax+by=$constant.

(*C*) ∞^∞ of families of ∞^1 curves: $\varphi(x)+\psi(y)=$constant.

The calculations for the invariants of the remaining groups will be much abbreviated. The subscripts i, j, k, l, will be used to denote the series of numbers running from 1, 2, 3, 4, respectively, to n.

(*A*) *The families of curves $x=$constant, $y=$constant, remain invariant.*

15. $\boxed{q,\quad yq}$

The differential equations

$$\Sigma i\frac{\partial f}{\partial y_i}=\Sigma\, i\, y_i\frac{\partial f}{\partial y}=0,$$

belonging to this group, evidently leave invariant

$$X_i \text{ and } \psi_k=(y_1-y_k):(y_1-y_2).$$

16. $\boxed{q,\quad yq,\quad y^2q}$

This is the general projective group in one variable, and leaves invariant x_i, and the cross-ratios of any four ordinates. We may determine these cross-ratios by substituting ψ_k of 15 in

$$\Sigma i y_i^2\frac{\partial f}{\partial y_i}=0,$$

giving

$$\Sigma k\psi_k(\psi_k-1)\frac{\partial f}{\partial \psi_k}=0,$$

and integrating,

$$\rho_l=\frac{\psi_l-1}{\psi_l}:\frac{\psi_3-1}{\psi_3}=\frac{y_2-y_l}{y_2-y_3}:\frac{y_1-y_l}{y_1-y_3}.$$

17. $\boxed{q,\quad yq,\quad p}$

This group evidently leaves invariant

$\psi_k=(y_1-y_k) : (y_1-y_2)$, and $\varphi_j=x_1-x_j$.

18. $\boxed{q, \quad yq, \quad y^2q, \quad p}$

The invariants for this group are clearly

$\rho_l=\dfrac{y_2-y_l}{y_2-y_3} : \dfrac{y_1-y_l}{y_1-y_3}$, as in 16 above, and

$\varphi_j=x_1-x_j$, as in 17.

19. $\boxed{q, \quad p, \quad xp+cyq}$

The solutions of the complete system corresponding to this group must be functions of $\psi_j=y_1-y_j$, $\varphi_j=x_1-x_j$, as shown by the first two members. These solutions, substituted in the last equation, give

$$\Sigma j\varphi_j\frac{\partial f}{\partial\varphi_j}+c\Sigma j\psi_j\frac{\partial f}{\partial\psi_j}=0,$$

with solutions

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}, \quad \sigma=\frac{(x_1-x_2)^c}{y_1-y_2}.$$

20. $\boxed{q, \quad yq, \quad p, \quad xp}$

The invariants here are

$$u_k=\frac{x_1-x_k}{x_1-x_2}, \quad v_k=\frac{y_1-y_k}{y_1-y_2}.$$

21. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp}$

By this group

$$\rho_l=\frac{y_2-y_l}{y_2-y_3} : \frac{y_1-y_l}{y_1-y_3}, \quad u_k=\frac{x_1-x_k}{x_1-x_2}$$

remain invariant.

22. $\boxed{q, \quad yq, \quad y^2q, \quad p, \quad xp, \quad x^2p}$

This six-parameter group leaves invariant the cross-ratios of any four abscissas and ordinates :

$$\rho_l=\frac{y_2-y_l}{y_2-y_3} : \frac{y_1-y_l}{y_1-y_3}, \quad \sigma_l=\frac{x_2-x_l}{x_2-x_3} : \frac{x_1-x_l}{x_1-x_3}.$$

23. $\boxed{p+q, \quad xp+yq, \quad x^2p+y^2q}$

$$\Sigma\left(\frac{\partial f}{\partial x_i}+\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i\frac{\partial f}{\partial x_i}+y_i\frac{\partial f}{\partial y_i}\right)=\Sigma\left(x_i^2\frac{\partial f}{\partial x_i}+y_i^2\frac{\partial f}{\partial y_i}\right)=0.$$

The solutions

$$\varphi_j = x_1 - x_j,\ \psi_j = y_1 - y_j,\ \sigma = x_1 - y_1$$

of the first equation, as new variables, give

$$\Sigma\left(\varphi_j \frac{\partial f}{\partial \varphi_j} + \psi_j \frac{\partial f}{\partial \psi_j}\right) + \sigma \frac{\partial f}{\partial \sigma} = \Sigma\left(\varphi_j^2 \frac{\partial f}{\partial \varphi_j} + \psi_j^2 \frac{\partial f}{\partial \psi_j}\right) + \sigma^2 \frac{\partial f}{\partial \sigma} = 0.$$

Selecting as solutions of the first of these new equations

$$u_k = \frac{\varphi_k}{\varphi_2},\quad v_k = \frac{\psi_k}{\psi_2},\quad \sigma_1 = \frac{\varphi_2}{\sigma},\quad \sigma_2 = \frac{\psi_2}{\sigma},$$

we have

$$\Sigma\left(u_k(1-u_k)\frac{\partial f}{\partial u_k} + v_k(1-v_k)\frac{\partial f}{\partial v_k}\right) + \sigma_1(1-\sigma_1)\frac{\partial f}{\partial \sigma_1} + \sigma_2(1-\sigma_2)\frac{\partial f}{\partial \sigma_2} = 0.$$

Hence, our invariants may be selected as the cross-ratios

$$\sigma_l = \frac{u_l(1-u_3)}{u_3(1-u_l)} = \frac{x_2 - x_l}{x_2 - x_3} : \frac{x_1 - x_l}{x_1 - x_3},\quad \rho_l = \frac{v_l(1-v_3)}{v_3(1-v_l)} = \frac{y_2 - y_l}{y_2 - y_3} : \frac{y_1 - y_l}{y_1 - y_3},$$

and the ratio

$$t_l = \frac{\sigma_1(1-\sigma_2)}{\sigma_2(1-\sigma_1)} = \frac{x_1 - x_2}{y_1 - y_2} : \frac{y_1 - x_2}{x_1 - y_2}.$$

(*B*) *All families of curves of the form* $ax + by =$ constant *remain invariant.*

24. $\boxed{q,\quad xp + yq}$

The first transformation of this group shows the differences $\psi_j = y_1 - y_j$ to enter in the final solutions. Hence the invariants may be written:

$$\rho_j = \frac{x_j}{x_1},\quad u_k = \frac{y_1 - y_k}{y_1 - y_2},\quad \sigma = \frac{y_1 - y_2}{x_1}.$$

25. $\boxed{p,\quad q}$

This group of translations leaves invariant

$$\varphi_j = x_1 - x_j,\ \psi_j = y_1 - y_j.$$

26. $\boxed{p,\quad q,\quad xp + yq}$

By this group any homogeneous function of the differences $x_1 - x_j$, $y_1 - y_j$ will remain invariant.

$$u_k = \frac{x_1 - x_k}{x_1 - x_2},\quad v_k = \frac{y_1 - y_k}{y_1 - y_2},\quad \sigma = \frac{x_1 - x_2}{y_1 - y_2}.$$

(*C*) *The totality of curves* $\varphi(x) + \psi(y) =$ constant *remains invariant.*

27. $\boxed{q}$

This group is the only one of the class, and evidently leaves invariant the abscissas x_i, and the differences

$$\psi_j = y_1 - y_j.$$

FEOREMA.

Hay numeros cuyo cuadrado se compond de la suma de una serie de cubos. j Cuales son estos numeros ? Fomernos la serie de los numeros naturales principrando por uno y vayamos hasta un numero calquiera, por ejemplo, hasta dies, asi.

1	2	3	4	5	6	7	8	9	10

Debajo de estos numeros pongamos sus cubos asi

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000

Sumemos estos cubos de manera que el primero venga á colo carse bajo el primer numero, la suma del primero y del segundo, bajo el segundo cubo : la suma, del primero segundo y tercero, bajo el tercero, y asi sucesuamente, tendremos

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025

y encontraremos que la suma de dichos cubos son cuadrados y sus raices son

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025
Raiz 1	3	6	10	15	21	28	36	45	55

Consideremos estas y raices veremos que son la suma de los térnimos de progresiones aritmeticas principrando por uno y segundo por los numeros naturales hasta cualquier numero que se desee y desde luego podremos formular el tevrema del modo siguiente.

Los cuadrados de los numeros que indican la suma de los términos deprogresiones aritmeticas, principrando por uno y siguiendo por los numeros naturales ; dichos cuadrados decimos, son iguales á la suma de los cubos de todos los numeros que entran en dicha progresion : ver : gr :

300. Es la suma de los terminos de una progresion aritmetica principrando por uno y acabando por veinte y cuatro y

300 elevado á la segunda potencia $=1^3+2^3+3^3+4^3 \ldots\ldots +24^3$.

J. M. Monsanto.

TRANSLATION.

THEOREM.

There are numbers whose square is composed of the sum of a series of cubes. Which are these numbers ?

Let us take the series of the primary numbers beginning with one and let us proceed as far as any number, for example, as far as ten, thus

1	2	3	4	5	6	7	8	9	10

Under these numbers let us place their cubes thus

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000

Let us add these cubes in such a way that the first comes to be placed under the first number, the sum of the first and the second under the second cube; the sum of the first, second and third, under the third cube, and so on, we shall have

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025

and we shall find that the sum of the said cubes are squares, and their roots are

1	2	3	4	5	6	7	8	9	10
1	8	27	64	125	216	343	512	729	1000
1	9	36	100	225	441	784	1296	2025	3025
1	3	6	10	15	21	28	36	45	55

Let us consider these roots and we shall see that they are the sum of the terms of an arithmetical progression beginning with one and continuing through the primary numbers as far as any number desired, and therefore we can formulate the theorem in the following manner:

The squares of the numbers that indicate the sum of the terms of an arithmetical progression, beginning with one and continuing through the primary numbers, said squares, we repeat, are equal to the sum of the cubes of all the numbers that enter in the said progression: for example 300^2. 300^2 is the sum of the terms of an arithmetical progression beginning with 1 and ending with 24, and 300 raised to the second power $= 1^3 + 2^3 + 3^3 + 4^3 \ldots\ldots + 24^3$.

J. M. Monsanto.

Note.—This rare bit of original mathematical work by J. M. Monsanto, with its translation, was sent to us by Dr. Halsted. Ed. F.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

100. Proposed by CHAS. C. CROSS, Libertytown, Md.

I bought stock at 4% discount, and sold it at 2½% premium, paying a brokerage in both cases of ¼%. If my net profits were $130, what was my investment? (Solve by Arithmetic).

I. Solution by W. F. BRADBURY, A.M., Head Master Cambridge Latin School, Cambridge, Mass., and M. E. GRABER, Tiffin, Ohio.

$4\% + 2\frac{1}{2}\% - \frac{1}{2}\%$ (brokerage) $= 6\%$.

That is, he made $6 on every $100 he invested. $130 \div 6 = 21\frac{3}{4}$.

He bought $21\frac{3}{4}$ shares of $100 each, or $\$2166\frac{2}{3}$ worth of stock.

But it cost him $3\frac{3}{4}\%$ below par.

$\$2166\frac{2}{3} \times 0.96\frac{1}{4} = \$2085.415$.

Solved in a similar manner by *P. S. BERG* and *W. H. DRANE*.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

"In both cases" is a dubious expression. If the brokerage in the two transactions was *together* ¼%, then the net gain on a dollar $= .04 + .02\frac{1}{2} - .00\frac{1}{4} = .06\frac{1}{4}$, and the investment $= \$130 \div .06\frac{1}{4} = \$2080$.

But if the brokerage in *each* transaction was ¼%, then the net gain on a dollar $= .04 + .02\frac{1}{2} - .00\frac{1}{2} = .06$; and the investment $= \$130 \div .06 = \$2166.66\frac{2}{3}$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Iowa.

Let $100 = 1$ share.

$4\% - \frac{1}{4}\% = 3\frac{3}{4}\%$; $100\% + 2\frac{1}{2}\% - \frac{1}{4}\% = 102\frac{1}{4}\%$. $100\% - 3\frac{3}{4}\% = 96\frac{1}{4}\%$.

$\$100 \div .96\frac{1}{4} = \$103.896$.

$\$103.896 \times 1.02\frac{1}{4} = \$106.23366$.

$\$106.23366 - \$100 = \$6.23366$, gain on one share.

$\$130 \div \$6.23366 = 20.85452$ shares.

$20.85452 \times \$100 = \$2085.452$, amount invested.

101. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A man gained $m = 3\%$ on his money, in July; and, in August, lost $n = 2\%$. What per cent. of his money July 1st is his money September 1st?

I. Solution by P. S. BERG, Principal of Schools, Larimore, N. D., and JOHN F. TRAVIS, Student in Ohio State University, Columbus, Ohio.

If 100% is his money on July 1st, then 103% is his money August 1st, and 98% of 103% or 100.94% is his money on September 1st.

Therefore his money September 1st is 100.94% of his money July 1st.

II. Solution by J. M. COLAW, A. M., Monterey, Va.; W. F. BRADBURY, A. M., Head Master Cambridge Latin School, Cambridge, Mass.; M. A. GRUBER, A. M., War Department, Washington, D. C.; WALTER HUGH DRANE, Graduate Student, Harvard University; and J. O. MAHONEY, B. E., M. Sc., Master of Mathematics and Science, Carthage Graded and High School, Carthage, Texas.

If p is his principal on July 1st, then $\frac{p(100+m)}{100}$ = his money on Aug. 1st.

Also, $\frac{p(100+m)}{100} - \frac{pn(100+m)}{10000}$, or $\frac{p(100-n)(100+m)}{10000}$ = his money on September 1st.

$$\therefore \frac{p(100-n)(100+m)}{10000} \div p = \frac{(100-n)(100+m)}{10000}, = \left[\frac{(100-n)(100+m)}{100}\right]\%.$$

Substituting numerical values for m and n, respectively, we get $\frac{98\times 103}{10000} = 1.0094, = 100.94\%$.

The result is independent of the value of p.

Also solved by *G. B. M. ZERR*.

ALGEBRA.

88. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School, Berea, O.

(1) The Indemnity Savings and Loan Company made two loans of \$1000 each to "A", one of its borrowers, under the following terms: In the first loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment to be considered as made on the first of the month in which the loan is made, and the 119 subsequent payments to be made on the first of each subsequent month; in the second loan "A" agrees to cancel the \$1000 by making 120 payments of \$13.50, the first payment being made on the first of the month following the loan, and the 119 subsequent payments being made on first of the subsequent months. Does the Company sustain any loss in earnings by the second loan over the first loan, and if so how much, and when is (or are) this loss (or these losses) sustained, the rate of interest in each case being considered as $10\frac{1}{2}\%$ per annum?

(2) Deduce a formula for each case of proposition (1) by means of which one can find the balance of the loan uncancelled at the end of *any* month, *if* the loan is fully cancelled in 120 payments.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1) Let P = principal, r = rate of interest, n = number of payments, x = each payment. Then it has been shown in several previous problems that

$$x = \frac{Pr(1+r)^n}{(1+r)^n - 1}.$$

In the first case "A" gets at once \$1000.00 − \$13.50 = \$986.50, and pays this amount in 119 monthly installments.

$\therefore P = 986.50$, $r = 10\frac{1}{2} \div 12 = 0.875\%$, $n = 119$.

$$\therefore x = \frac{986.50\times .00875(1.00875)^{119}}{(1.00875)^{119}-1}.$$

x = \$13.375 or \$0.125 less than \$13.50.

In the second case $P = 1000$, $r = .00875$, $n = 120$.

$\therefore x = \dfrac{1000 \times .00875(1.00875)^{120}}{(1.00875)^{120}-1}$.

$\therefore x = \$13.494$.

In the second case the true interest is realized. $\therefore$ $10\frac{1}{2}$ per cent. is charged. In the first case more than $10\frac{1}{2}$ per cent. is charged. $\therefore$ In the first case the borrower by paying \$13.50 pays $12\frac{1}{2}$ cents more each payment than if he paid $10\frac{1}{2}$ per cent. interest per annum.

(2) Pr=interest. $\therefore$ $x-Pr$=principal paid at the first payment.

$P(1+r)-x$=new principal; $Pr(1+r)-rx$=second interest.

$\therefore$ $x(1+r)-Pr(1+r)=(x-Pr)(1+r)$=principal paid at second payment.

Similarly, $(x-Pr)(1+r)^{m-1}$=principal paid at the mth payment.

$\therefore$ $x-Pr$=first term, $1+r$=ratio, m=number of terms.

$\therefore S = \dfrac{(x-Pr)[(1+r)^m-1]}{r}$.

Unpaid principal at the end of m payments is $P-S$.

Let $x=13.50$, $P=1000$, $r=.00875$, $m=60$.

$\therefore$ $S=\$372.72$.

$\therefore$ $P-S=\$1000-\$372.72=\$627.28$ unpaid.

II. Partial Solution by A. H. BELL, Hillsboro, Ill.

(1) The first loan of \$1000 for 10 years at $10\frac{1}{2}$ per cent. amounts to \$2714.06.....(1).

The second loan of \$986.50 for $9\frac{11}{12}$ years at $10\frac{1}{2}$ per cent. amounts to \$2655.24.....(2).

Let the common form of the amount of an annuity be:

$$\text{Amount annuity} = A\left(\frac{R'^n-1}{r'}\right).....(3),$$

in which $A=\$13.50$, n=number of payments, R'=the amount of \$1.00 for one month at $10\frac{1}{2}$ per cent.=\$1.00875, r'=interest for one month at $10\frac{1}{2}$ per cent.=.00875.

$\therefore$ The amount of the annuity \$13.50 for 120 months is \$2847.26.. .(4); the amount of the annuity \$13.50 for 119 months is \$2809.18 ....(5).

(4)$-$(1)=\$133.20 is the profit in the first transaction. ....(6);

(5)$-$(2)=\$153.94 is the profit in the second transaction.....(7).

Since the second loan of \$1000 is immediately reduced by the first payment of \$13.50, no comparison can be made with the first loan, for the second actual loan of \$986.50 pays a greater amount of annuity on the dollar and consequently shows a greater amount earned as seen in (7).

III. Solution by FREMONT CRANE, Sand Coulee, Mont.

(a) First payment could draw interest for 120 months; last payment could draw interest for 1 month. $\therefore$ average time=$60\frac{1}{2}$ months.

(b) First payment would draw interest for 119 months; last payment would draw interest for 0 months. $\therefore$ average time is $59\frac{1}{2}$ months.

$\therefore$ (a) would be the better investment.

$120 \times \$13.50 = \$1620$; $\$1620 \times .10\frac{1}{2} \times 1\frac{1}{2} = \$14.175$ = profit of first investment over second.

2. (a) Let $\$13.50 = p$, $.00875 = r$, $\$1000 = a$; then $(a-p)$ = amount due first of first month; $(a-p)(1+r)-p$ = amount due first of second month; $[(a-p)(1+r-p)](1+r)-p$ = amount due first of third month; $\{[(a-p)(1+r)-p)](1+r)-p\}(1+r)-p$ = amount due first of fourth month.

$\therefore$ for any month, as 10,

$\{[(a-p)(1+r)-p](1+r)-p](1+r)-p\}$ $[(a-p)(1+r)-p)(1+r)-p^7(1+r)-p]$.

2. $a(1+r)-p$ at end of first month; $[a(1+r)-p](1+r)-p$ at end of second month; $\{[a(1+r)-p](1+r)-p\}(1+r)-p$ at end of third month. At end of tenth month, $\{\{[a(1+r)-p](1+r)-p\}^8(1+r)-p\}$.

The solution by Dr. Zerr seems to us to be a proper disposition of the problem, but we publish the other solutions for comparison.

Reply to Note of Dr. Drummond on No. 78.—Its one *real* point is, that my solution, on page 43, line 5, omits to denote $-xy=-6$, as $-x_1y_1=-6$, (which please correct). Thus result $x^4-8x^2+16=0$; $y^4-18y^2+81=0$, whose roots are: $x=2$ to $y=3$; $x_a=-2$ to $y_a=-3$, of xy in (I)(II); and also $x_1=2_1$ to $y_1=3_1$; $x_2=-2_1$ to $y_2=-3_1$ (by last two clauses on page 44), the factors of $-x_1y_1=-6_1$ in parity to $x_{11}y_{11}=35$.....(D) on page 44. Or if the objector insist,—of $x_1y_1=6_1$ and $x_ay_a=6_a$ reading then $xy=6$ and $x_ay_a=6_a$. No other error is specified or shown. As the equations cited do not occur in my solution,—for his $x^2=36$ is quadratic while $x^2y^2=36$ from (I)(II) is bi-quadratic; I need only say that the roots of his $x^4=1296$ are 6; -6, $6_1/(-1)$; $-6_1/(-1)$, not ± 6; ∓ 6. Roots $x^4=16$ are 2; -2; $2_1/(-1)$; -2_1 (-1) and not ± 2; ∓ 2 as he mistakes me to mean. J. M. BOORMAN.

GEOMETRY.

99. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.**

Find the locus of the vertices of all right cones which have the same given ellipse as a base.

Solution by the PROPOSER.

Any generator, $x=a'z+\alpha$.....(1), $y=b'z+\beta$.....(2), must satisfy

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.....(3), and $z=0$.....(4), giving $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=1$.....(5).

If the vertex be (x', y', z'), (1) and (2) must be satisfied, and we have $x'=a'z'+\alpha$.....(6), $y'=b'z'+\beta$.....(7).

Eliminating a', b', α, β from (1), (2), (6), (7),

$$\frac{1}{a^2}(z'x-x'z)^2+\frac{1}{b^2}(z'y-y'z)^2=(z'-z)^2 \quad \ldots\ldots \quad (8),$$

the equation to a cone having (x', y', z') for vertex, and (3) for base.

(8) is $\frac{z'^2}{a^2}x^2 + \frac{z'^2}{b^2}y^2 + \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1\right)z^2 - \frac{2y'z'}{b^2}yz - \frac{2x'z'}{a^2}xz + 2z'z - z'^2 = 0 \ldots (9).$

The conditions for (9) to be a cone of revolution are $x'=0$,

$$\frac{z'^2}{a^2} - \frac{y'^2}{b^2 - a^2} = 1 \ldots\ldots (10),$$

an hyperbola for the required locus.

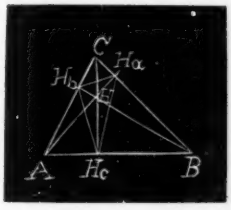
Also solved by *G. B. M. ZERR*.

100. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

O, O_1, O_2, O_3 are the centers of the inscribed and three escribed circles of a triangle ABC. Prove $AO.AO_1.AO_2.AO_3 = AB^2.AC^2$.

I. Solution by the PROPOSER.

Consider the ex-central triangle ABC as the original triangle. Then $H_aH_bH_c$ is the pedal triangle, and the incenter O becomes the orthocenter H.



Hence we have to prove $AH_c \times BH_c \times CH_c \times HH_c = H_aH_c^2 \times H_bH_c^2$.

We readily find by trigonometry that

$$AH_c = \frac{b^2 + c^2 - a^2}{2c}, \quad BH_c = \frac{a^2 + c^2 - b^2}{2c},$$

$$CH_c = \frac{1}{2c}\sqrt{[4a^2c^2 - (a^2 + c^2 - b^2)^2]} = \frac{2\triangle}{c}, \quad HH_c = \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{8c\triangle},$$

$$H_aH_c = \frac{b(a^2 + c^2 - b^2)^2}{2ac}, \quad H_bH_c = \frac{a(b^2 + c^2 - a^2)}{2bc}.$$

Substituting in the problem we have

$$\frac{b^2 + c^2 - a^2}{2c} \times \frac{a^2 + c^2 - b^2}{2c} \times \frac{2\triangle}{c} \times \frac{(b^2 + c^2 - a^2)(a^2 + c^2 - b^2)}{8c\triangle}$$

$$= \frac{b^2(a^2 + c^2 - b^2)^2}{4a^2c^2} \times \frac{a^2(b^2 + c^2 - a^2)^2}{4b^2c^2}.$$

Since these equations cancel, the proposition is proved.

Mr. Cross should have been credited with solutions of problems 96 and 98 in Geometry, and 100 and 101 in Arithmetic.

II. Solution by WALTER H. DRANE, Graduate Student at Harvard University, and J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the given triangle, O, O_1, O_2, O_3, the centers of the inscribed and the three escribed circles. Then O_1BO_2, O_1AO_3, O_2CO_3, and AOO_2 are straight lines; also OB, OA, OC are each perpendicular to O_1BO_2, O_1AO_3, and O_2CO_3, respectively. In triangles AOC and BAO_2, $\angle OAC = BAO_2$ and $\angle OCA = \angle BO_2A$ since we have $\angle OCA = 90° - \angle ACO_3 = 90° - \frac{1}{2}(\angle CAB + \angle CBA) = 90° - (\angle OAB + OBA) = 90° - [180° - (\angle O_1AB + \angle O_1BA)] = 90° - \angle O = \angle O_1O_2A$.

$\therefore$ triangles AOC and ABO_2 are similar, and we have

$$AO : AB :: AC : AO_2 \quad \ldots\ldots\ldots (1).$$

Again in triangles O_1BA and AO_3C, $\angle O = \angle ACO_3$ and $\angle O_3AC = \angle O_1AB$. Hence triangles O_1BA and AO_3C are similar, and we have,

$$AO_1 : AC :: AB : AO_3 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying (1) by (2) $AO.AO_1 : AB.BC : AB.AC : AO_2.AO_3$.

$\therefore AO.AO_1.AO_2.AO_3 = AB^2.AC^2$. Q. E. D.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In the figure of the last solution, draw the lines OD and O_1D_1 perpendicular to AC. Then $AO = \sqrt{(OD^2 + AD^2)} = \sqrt{(r^2 + r^2\cot^2 \frac{1}{2}A)} = r\operatorname{cosec} \frac{1}{2}A$.

Similarly, $AO_1 = r_1 \operatorname{cosec} \frac{1}{2}A$.

$AO_2 = \sqrt{(OO_2^2 - AO^2)} = AO\sqrt{[(OO_2^2/AO^2) - 1]} = AO\cot \frac{1}{2}C$.

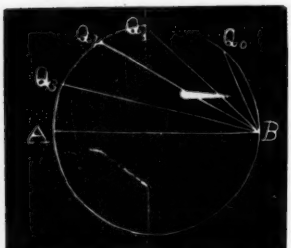
Similarly, $AO_3 = AO\cot \frac{1}{2}B$.

$$\begin{aligned}\therefore AO.AO_1.AO_2.AO_3 &= r^3 r_1 \operatorname{cosec}^4 \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C \\ &= (s-a)(s-b)(s-c)s \operatorname{cosec}^4 \tfrac{1}{2}A \tan^2 \tfrac{1}{2}A \\ &= s(s-a)(s-b)(s-c)/\sin^2 \tfrac{1}{2}A \cos^2 \tfrac{1}{2}A \\ &= 4s(s-a)(s-b)(s-c)/\sin^2 A = 4S^2/\sin^2 A.\end{aligned}$$

But $\sin A = 2S/bc = 2S/AB.AC$. $\therefore AO.AO_1.AO_2.AO_3 = AB^2.AC^2$.

Also solved by *ELMER SCHUYLER*.

101. Proposed by E.W. MORRELL, A.M., Late Professor of Mathematics, Montpelier Seminary, Montpelier, Vt.



AB is the diameter of a circle and Q_0 any point on the circumference; Q_1, Q_2, Q_3.... are the points of bisection of the arcs AQ_0, AQ_1, AQ_2.... Prove that BQ_1, BQ_2, BQ_3 $\ldots . BQ_n = OA^n.(AQ_0/AQ_n)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A.M., Hagerstown, Md., and ELMER SCHUYLER, High Bridge, N. J.

Let O be the center of the circle.

$\angle ABQ_0 = \theta$.

$\therefore BQ_1 = AB\cos \frac{1}{2}\theta$, $BQ_2 = AB\cos(\theta/2^2)$.

$BQ_3 = AB\cos(\theta/2^3)$, $BQ_n = AB\cos(\theta/2^n)$.

$AQ_0 = AB\sin\theta = 2AB\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = 2^2 AB\sin(\theta/2^2)\cos(\theta/2^2)\cos\tfrac{1}{2}\theta$
$= 2^n AB\sin(\theta/2^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\ldots\cos(\theta/2^n).$

$AQ_n = AB\sin(\theta/2^n).$

$\therefore (AQ_0/AQ_n) = 2^n\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\ldots\cos(\theta/2^n).$

$BQ_1.BQ_2.BQ_3\ldots\ldots BQ_n = (AB^n)\cos\tfrac{1}{2}\theta\cos(\theta/2^2)\ldots\ldots\cos(\theta/2^n)$
$= (\tfrac{1}{2}AB)^n(AQ_0/AQ_n) = (AO)^n(AQ_0/AQ_n).$

CALCULUS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Investigate value of $\left(\frac{\tan x}{x}\right)^{1/x^n}$ where x is 0 and n has consecutive values 1, 2, 3, 4, ... Is there any law governing the different results? When $n=1$, result is 1; when $n=2$, result is $e^{\frac{1}{3}}$; $n=3$, gives ∞, etc.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$$\left(\frac{\tan x}{x}\right)^{1/x^n} = e^{(1/x^n)\log[(\tan x)/x]} = y.$$

Limit of $\frac{\log\tan x - \log x}{x^n}$ = limit of $\frac{\cot x\sec^2 x - (1/x)}{nx^{n-1}}$ = limit of $\frac{2x - \sin 2x}{nx^n\sin 2x}$,

but $\sin 2x = 2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \frac{128x^7}{5040}$, etc., $= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \frac{4x^7}{45} +$, etc.

$$\therefore \frac{2x - \sin 2x}{nx^n\sin 2x} = \frac{2x - 2x + \frac{4x^3}{3} - \frac{4x^5}{15} + \frac{4x^7}{45} -, \text{ etc.}}{nx^{n+1}\left(2 - \frac{4x^2}{3} + \frac{4x^4}{15} - \frac{4x^6}{45} +, \text{ etc.}\right)} = \frac{30 - 6x^2 + 2x^4}{nx^{n-2}(45 - 30x^2 + 6x^4)},$$ approximately, $= \frac{2}{3nx^{n-2}} + \frac{14}{45nx^{n-4}} +$, etc., $= S.$

When $n=1$, $S=0$ for $x=0$.
When $n=2$, $S=\frac{1}{3}$ for $x=0$.
When $n=3, 4, 5$, etc., $S=\infty$ for $x=0$.
$\therefore$ When $n=1$, $y=e^0=1$.
When $n=2$, $y=e^{\frac{1}{3}}$.
When $n=3, 4, 5$, etc., $y=e^\infty=\infty$.

Also solved by *ELMER SCHUYLER*, whose solution has been accidentally misplaced, and hence does not appear in this issue.

79. Proposed by GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

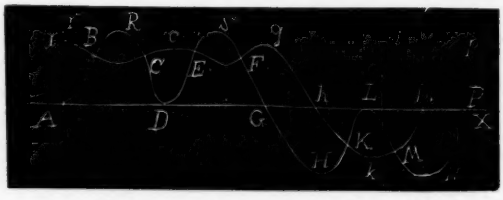
Find the area included between $y=\sin^{\pi} x+\cos^{e}x$; $y=\pi e(\sin^{\pi} x\cos^{e}x)$ and the length of its boundary, true to six decimal places, when $\pi=3.14159$, $e=2.7182$.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

When $x=0°$, $\sin^{\pi} x=0$, $\cos^{e}x=1.0000$.
When $x=30°$, $\sin^{\pi} x=.1133$, $\cos^{e}x=.6764$.
When $x=45°$, $\sin^{\pi} x=.3366$, $\cos^{e}x=.3899$.
When $x=60°$, $\sin^{\pi} x=.8006$, $\cos^{e}x=.1520$.
When $x=90°$, $\sin^{\pi} x=1.0000$, $\cos^{e}x=0$.
Since e is even, $\cos^{e}x$ is always positive ; hence the following coördinates :

	$y=\sin^{\pi} x+\cos^{e}x$	$y=\pi e(\sin^{\pi} x\cos^{e}x)$
x	y	y
$0°=.0$	1.0000	0
$30°=.5236$	.7897	.6541
$45°=.7853$	.7265	1.1207
$60°=1.0471$	.9526	1.0392
$90°=1.5707$	1.0000	0
$120°=2.0942$	.9526	1.0392
$135°=2.3560$	.7265	1.1207
$150°=2.6178$	.7897	.6541
$180°=3.1414$	1.0000	0
$210°=3.6650$	.5631	$-.6541$
$225°=3.9268$	.0533	-1.1207
$240°=4.1886$	$-.6480$	-1.0392
$270°=4.7121$	-1.0000	0
$300°=5.2360$	$-.6480$	-1.0392
$315°=5.4978$	.0533	-1.1207
$330°=5.7596$	.5631	$-.6541$
$360°=6.2828$	1.0000	0

The curves drawn from these coördinates are somewhat like the figure, $ABRCDESFGHKLMNP$ corresponding to $y=\pi e(\sin^{\pi} x\cos^{e}x)$; $aBCEFghKkMlp$ corresponding to $y=\sin^{\pi} x+\cos^{e}x$.



The curves are indefinite in length. The areas included between the curves from $x=0°$ to $x=360°$ are $AaB+BRC+CDEc+ESF+FGHKhg+LMkK+PNMlp$. The length of the boundary is the whole length of both curves. The whole area common to both curves is infinite. The area above is but the area for one revolution. Area BRC=area ESF.

area AaB + area $PNMlp$ = area $FGHKhg$. The intersections are $x=32^\circ\ 48'$, $x=61^\circ$, $x=119^\circ$, $x=147^\circ\ 12'$, $x=244^\circ\ 22'$, $x=295^\circ\ 38'$. The integrations are exceedingly tedious, but can be performed. If 3.1416 had been used for π the curves for one revolution would have consisted of four parts each equal to $aBCc$ and $ABRCD$.

MECHANICS.

Criticism on Professor Zerr's Solution of Problem 67, Mechanics, by J. M. ARNOLD, Crompton, R. I.

I wish to take exception to Professor Zerr's solution of No. 63 Mechanics, in the May number. The preliminary reasoning and the diagram are correct, but when he proceeds to find the required angles he commences with the assumption "The $\angle ABC = \angle CDE$ and the $\angle BAC = \angle CED$." This is wrong as it can be easily shown that these angles are not equal. Therefore his result must be in error. I have not had time to solve the problem correctly, but I think it leads to very complicated equations.

70. Proposed by CHARLES E. MEYERS, Canton, Ohio.

A homogeneous sphere, radius r, having an angular velocity ω, gradually contracts by cooling. What will be the angular velocity at the instant the radius becomes $\frac{1}{2}r$?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let m = the constant mass ; r, $\frac{1}{2}r$ the original and final radii ; ω', the required angular velocity ; k, k' the radii of gyration corresponding.

The moment of angular momentum remaining constant,

$$mk^2\omega = mk'^2\omega' \ldots\ldots (1).$$

But $k^2 = \frac{1}{2}r^2$, $k'^2 = \frac{1}{2}(\frac{1}{2}r)^2 = \frac{1}{8}r^2$, (1) plainly gives $\omega' = 4\omega$.

Also solved in the same manner by *G. B. M. ZERR.*

71. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Three men own a sphere of gold the density of which varies as the square of the distance from the center. If two segments be cut off each one inch from the center of the sphere it will be divided into three parts of equal value. Determine the diameter of the sphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ρ = density = r^2 in this case, a = radius. Then the mass of each segment cut off is

$$M = \int_1^a \int_0^{2\pi} \int_0^{\cos^{-1}(1/a)} \rho r^2 \sin\theta d\theta d\varphi dr = \int_1^a \int_0^{2\pi} \int_0^{\cos^{-1}(1/a)} r^4 \sin\theta dr d\varphi d\theta$$

$$= \frac{2\pi}{5a}(a^5 - 1)(a - 1).$$

The mass of the remaining part is

$$M_1=2\int_0^1\int_0^{2\pi}\int_{\cos^{-1}(1/a)}^{\frac{1}{2}\pi} r^4\sin\theta\,dr\,d\varphi\,d\theta=\frac{4\pi}{5a}.$$

But $M=M_1$.

$\therefore \frac{4\pi}{5a}=\frac{2\pi}{5a}(a^6-a^5-a+1).$

$\therefore a^6-a^5-a-1=0.$

$\therefore a^3-(1/a^3)=a^2+(1/a^2)$, or $t^3-t^2+3t-2=0$ where $t=a-(1/a)$.

Let $t=s+\frac{1}{3}$, $\therefore s^3+\frac{8}{3}s=\frac{29}{27}$.

$\therefore s=.381891,\ t=.715224.$

$a=\frac{1}{2}[t+\sqrt{(4+t^2)}]=1.4196$ inches nearly.

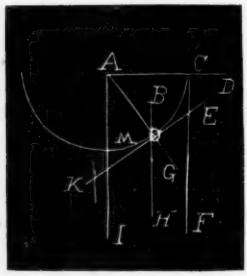
Similarly solved by *J. SCHEFFER*, his answer being 3.7462 inches.

72. Proposed by REV. A. L. GRIDLEY, Pastor of First Congregational Church, Kidder, Mo.

Prove that the motion of a ball falling through the earth influenced by gravity alone would be similar to the motion of a pendulum.

I. Solution by the PROPOSER.

It is an established fact that if a body should fall through an opening through the center of the earth, the net force drawing it toward the center is as the remaining distance to be traversed.



The pendulum ball follows the same law. It is evident that when the rod is horizontal all the force of gravity is exerted upon it to move it along its vertical tangent. When the rod is vertical, there is no net available force to move it from its position.

Making it general, by resolution of forces, the net force of gravity available to move the ball along the tangent EK is, in the figure, equal to the $\measuredangle AMB$. But this angle$=\measuredangle IAM$, as lines AI and BH are parallel and are cut by line AG. But $\measuredangle IAG$ represents the remaining distance for the ball to pass through. Thus the law in both cases is the same.

Inference. 1st. To vibrate in the same time the whole arc—semi-circle—of the pendulum must equal the diameter of the earth.

2d. Vibrating past the center for any distance would require the same time as from surface to surface.

3d. The net available force acting is in inverse proportion to the distance passed through.

Similarly solved by *ALOIS F. KOVARIK*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; CHARLES C. CROSS, Libertytown, Md.; CHARLES E. MEYERS, Canton, Ohio, and J. SCHEFFER, A. M., Hagerstown, Md.

The equation of motion is $d^2x/dt^2=-gx/R$. Where g=gravity=32½ feet, R=3963 miles=radius of earth. Multiplying by $2dx$ and integrating, we get $(dx/dt)^2=-gx^2/R+C$.

Let $x=a$ when $dx/dt=0$. Then $C=ga^2/R$.

$\therefore (dx/dt)^2=(g/r)(a^2-x^2)$.....(1).

$\therefore dt=\sqrt{(R/g)}.dx/\sqrt{(a^2-x^2)}$.

$\therefore t=\sqrt{(R/g)}\cos^{-1}(x/a)+C_1$.

Since $t=0$ when $x=a$, $C_1=0$.

$\therefore t=\sqrt{(R/g)}\cos^{-1}(x/a)$.....(2).

Equation (2) represents simple harmonic motion. From (1), velocity is zero when $x=\pm a$ and a maximum when $x=0$.

When $x=0$, $t=\frac{1}{2}\pi\sqrt{(R/g)}$. This time is the same for any value of a between 0 and R.

$2t$=time of complete vibration=$\pi\sqrt{(R/g)}$.

$2t$=42 minutes, 12 seconds.

Also solved by J. H. DRUMMOND.

DIOPHANTINE ANALYSIS.

69. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Two right angled triangles have the same base which is a mean proportional between the two perpendiculars: find a general solution, that will give integral values for all the sides of both triangles.

I. Solution by the PROPOSER.

Let x=the base, y=the perpendicular of one triangle, and z=that of the other triangle. Then $x^2=yz$.....(1), and $x^2+y^2=\square$.....(2), and $x^2+z^2=\square$ ..... (3). Substituting yz for x^2 in (2) and (3), we have $zy+y^2=\square$.....(4), and $zy+z^2=\square$.....(5). Take $y=mz$ and substituting in (4) and (5), and reducing, we have $m+1=\square=$, say, p^2; and $m=p^2-1$.

$m(m+1)=\square$.....(6), substituting the value of m in (6), we have $p^2(p^2-1)=\square$. Hence we have to make $p^2-1=\square=$, say, $(q-p)^2$. Hence $p=(q^2+1)/2q$ and $m=[(q^2-1)/2q]^2$; and $y=z[(q^2-1)/2q]^2$. $x=\sqrt{(yz)}=z[(q^2-1)/2q]$; to obtain integral values take $z=4q^2$, and we have $x=2q(q^2-1)$, $z=4q^2$ and hypotenuse$=2q(q^2+1)$, and $x=2q(q^2-1)$, $y=(q^2-1)^2$ and hypotenuse$=(q^2+1)(q^2-1)$, in which q may be any number >1.

If the value of q is an odd number, it is manifest that one fourth of each of the foregoing values will be integral, and we shall have smaller values.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let p, h be the perpendiculars; c, d the hypotenuses; k the common base.

Then $p^2+k^2=c^2$, $h^2+k^2=d^2$, $ph=k^2$.
Let $p=a^3+a^2b$, $h=ab^2+b^3$.
$\therefore k^2=a^2b^2(a+b)^2$.
$\therefore a^2(a+b)^2(a^2+b^2)=c^2$, $b^2(a+b)^2(a^2+b^2)=d^2$.
Let $a=m^2-n^2$, $b=2mn$, $m>n$.
$\therefore p=(m^2-n^2)^2(m^2-n^2+2mn)$,
$h=4m^2n^2(m^2-n^2+2mn)$,
$k=2mn(m^2-n^2)(m^2-n^2+2mn)$,
$c=(m^2-n^2)(m^2-n^2+2mn)(m^2+n^2)$,
$d=2mn(m^2-n^2+2mn)(m^2+n^2)$.
Let $m=2$, $n=1$.
$\therefore p=63$, $h=112$, $k=84$, $c=105$, $d=120$.

III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; SYLVESTER ROBINS, North Branch Depot, N. Y., and CHARLES C. CROSS, Libertytown, Md.

(*a*) Take $2mn$, m^2-n^2, and m^2+n^2, the general expressions for the sides of right triangles. Now multiply these sides, respectively, by $r(m^2-n^2)$ and $2mnr$ and we obtain the respective sides of the two required triangles :

(1) $2mnr(m^2-n^2)$, $r(m^2-n^2)^2$, $r(m^2-n^2)(m^2+n^2)$,

(2) $2mnr(m^2-n^2)$, $r(2mn)^2$, $2mnr(m^2+n^2)$,

in which m, n and r may be any integers, $m>n$.

Take $n=r=1$ and $m=2$. Then the sides of the two triangles are, respectively, 12, 9, 15, and 12, 16, 20 ; for $12^2=9\times16$.

$m=2$, $n=1$, $r=3$, gives 36, 27, 45, and 36, 48, 60.

$m=r=3$, $n=2$, gives 180, 75, 195, and 180, 432, 468.

[GRUBER, CROSS.]

(*b*) Multiply the sides of any integral right triangle by each leg, or the same number of times each leg, respectively, of the triangle, and the two results will be the respective sides of the two required triangles.

Take the sides 8, 15, 17. Multiply, respectively, by the legs 8 and 15, and we obtain the triangles 120, 64, 136, and 120, 225, 255.

Take the sides 5, 12, 13. Multiply, respectively, by 2×5 and 2×12, and we obtain the triangles 120, 50, 130, and 120, 288, 312.

[ROBINS, GRUBER.]

IV. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, Ohio.

Let $2mn$, m^2-n^2, and m^2+n^2; $2m'n'$, $m'^2-n'^2$ and $m'^2+n'^2$ be the pair of right triangles fulfilling the conditions

$$2mn=2m'n' \ldots\ldots (1),$$

$$\text{and } 4m^2n^2=4m'^2n'^2=(m^2-n^2)(m'^2-n'^2) \ldots\ldots (2).$$

From (2), $m'^2-n'^2=\dfrac{4m^2n^2}{m^2-n^2}$ ..... (3).

Squaring (3), $m'^4-2m'^2n'^2+n'^4=\dfrac{16m^4n^4}{(m^2-n^2)^2}$ ..... (4).

We have $4m'^2n'^2=4m^2n^2$ .....(5).

Adding (4) and (5), $(m'^2+n'^2)^2=4m^2n^2\left(\frac{4m^2n^2}{(m^2-n^2)^2}+1\right)$

$$\text{or } m'^2+n'^2=\frac{2mn(m^2+n^2)}{m^2-n^2} \ldots\ldots (6).$$

m and n must be selected so as to make (3) and (6) integral.

70. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadrates, and show that 61×200^3 is the sum of ten cube numbers, and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

Comment by the PROPOSER.

Dr. Artemas Martin has written me that he knows of no method of separating a given number into squares, cubes and other powers, except *trial*.

I take the following numbers from *The Mathematical Magazine*.

$61\times200^3=1^3+2^3+4^3+6^3+15^3+46^3+60^3+270^3+500^3+700^3$.

$844933=27^4+20^4+16^4+15^4+12^4+10^4+8^4+7^4+3^4+2^4+1^4$,

$=25^4+24^4+16^4+15^4+8^4+6^4+5^4+4^4+3^4+2^4+1^4$,

$=25^4+24^4+15^4+14^4+12^3+10^4+7^4+4^4+3^4+2^4+1^4$,

$=25^4+22^4+18^4+15^4+14^4+11^4+10^4+6^4+3^4+2^4+1^4$,

$=25^4+21^4+18^4+17^4+14^4+12^4+10^4+6^4+5^4+4^4+1^4$,

$=25^4+21^4+18^4+16^4+15^4+12^4+10^4+9^4+6^4+3^4+2^4$,

$=25^4+21^4+18^4+15^4+14^4+13^4+12^4+10^4+8^4+7^4+2^4$,

$=25^4+20^4+19^4+18^4+14^4+10^4+9^4+7^4+6^4+4^4+3^4$,

$=25^4+20^4+19^4+16^4+14^4+13^4+11^4+10^4+9^4+4^4+2^4$,

$=24^4+23^4+18^4+15^4+14^4+12^4+10^4+9^4+6^4+5^4+3^4$,

$=24^4+21^4+20^4+16^4+15^4+13^4+10^4+7^4+6^4+4^4+1^4$,

$=24^4+21^4+20^4+16^4+15^4+12^4+11^4+8^4+7^4+5^4+2^4$,

$=24^4+21^4+19^4+18^4+15^4+12^4+10^4+6^4+5^4+3^4+2^4$.

MISCELLANEOUS.

64. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

How many bushels of wheat will a conical bin 8 feet in diameter at base and 12 feet high hold, if part of the bin is cut off by a plane parallel to the side and passing through the center of the base?

Solution by the PROPOSER.

Let ABC be the cone, DF the plane.

Let $BD=c$, $\tan DBC=\cot FDC=n$, $DC=R$.

$\therefore x^2+z^2=n^2(c-y)^2$ is the equation to the cone.

$x=ny$ is the equation to the plane.

$V=$volume of cone cut off by plane.

The limits of x are $ny=x_2$ and $n(c-y)=x_1$; of y, 0 and $\frac{1}{2}c$.

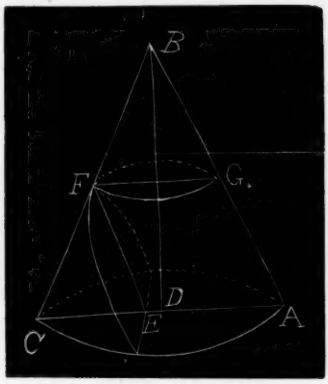
$$\therefore V=2\int_0^{\frac{1}{2}c}\int_{x_2}^{x_1}\sqrt{[n^2(c-y)^2-x^2]}\,dy\,dx$$

$$=\int_0^{\frac{1}{2}c}\{\tfrac{1}{2}\pi n^2(c-y)^2-n^2(c-y)^2\sin^{-1}\left(\frac{y}{c-y}\right)$$

$$-ny\sqrt{[n^2(c-y)^2-n^2y^2]}\}\,dy$$

$$=\tfrac{1}{8}\pi n^2c^3-\tfrac{2}{9}n^2c^3=\tfrac{1}{18}n^2c^3(3\pi-4)$$

$$=\tfrac{1}{18}R^2c(3\pi-4).$$



But $c=12$, $R=4$.

$\therefore V=32\pi-\frac{128}{3}$.

Volume of cone$=\frac{1}{3}\pi R^2c=64\pi$.

$\therefore$ Required vol.$=64\pi-(32\pi-\frac{128}{3})=32\pi+\frac{128}{3}$

$=143.1978$ cubic feet,

$=115.07$ bushels.

Also solved by *P. S. BERG* and *C. C. CROSS*.

[NOTE.—In the figure, the point *E* should coincide with *D*. ED. F.]

65. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

No solution has yet been received.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

102. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A's age is to B's as 2:3. 20 years from now their ages will be to each other as 4:5. What are their ages, respectively ?

103. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Find proceeds of a note discounted at a bank for 10 years at 10%. What is the meaning of the result ?

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

ALGEBRA.

92. Proposed by ELMER SCHUYLER, High Bridge, N. J.

Given $x^2-yz=1$; $y^2-xz=2$; $z^2-xy=3$. Find x, y, and z.

93. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Given $x^x+y^y=285$, and $y^x-x^y=14$, to find the values of x and y. [From *Bonnycastle's Algebra*, 1841.]

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

GEOMETRY.

108. Proposed by NELSON L. RORAY, Bridgeton, N. J.

ABC is a triangle. O_1, O_2, O_3 centers of escribed circles. Prove altitudes of triangle $O_1O_2O_3$ are concurrent at center of inscribed circle.

109. Proposed by CHARLES CARROLL CROSS, Libertytown, Md.

Two circles, radii in ratio 3:1, centers A and O_1 respectively, are drawn tangent externally to each other and internally to a given circle O, and on the same diameter ; O_2 and O_2' are drawn tangent internally to O and externally to A and O_1 ; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2 ; O_3 and O_3' are drawn tangent internally to O and externally to A and O_2, A and O_2', respectively ; and so on. Prove O_4, O, O_4' ; O_5, A, O_5' ; O_9, A, O_3' and O_{10}, O, O_2' are collinear. [The letters apply to the centers of the circles.]

110. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

If the three face angles of the vertical triedral angle of a tetraedron are right angles, and the lengths of the lateral edges are represented by a, b, and c, and of the altitude by p, then $1/p^2=1/a^2+1/b^2+1/c^2$. [*Chauvenet's Geometry.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

CALCULUS.

83. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

From a given point, P, in the base AB of a triangle, to inscribe in the latter the minimum triangle, if its angle at P is given.

84. Proposed by C. HORNUNG, A. M., Professor of Mathematics, Heidelberg University, Tiffin, Ohio.

Find the equation of the curve upon which a given ellipse must roll in order that one of its foci may describe a straight line.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

MECHANICS.

77. Proposed by ELMER SCHUYLER, High Bridge, N. J.

At what elevation must a shell be projected with a velocity of 400 feet that it may range 7500 feet on a plane which descends at an angle of 30°?

78. Proposed by ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Iowa.

A cone and a cylinder having equal heights and equal circular bases are filled with

water; if they have equal holes in the bases, respectively, how many times as long will it take the cylinder to empty as the cone?

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

DIOPHANTINE ANALYSIS.

76. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

It is required to find four positive numbers, such that if each be diminished by twice the cube of their sum the four remainders will be rational cubes.

77. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find (1) three consecutive numbers whose sum is a cube, and (2) three consecutive numbers the sum of whose cubes is a cube.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, 65 Hammond Street, Cambridge, Mass.

Prove $\tan^{-1}x=\frac{1}{2i}\left(\log\frac{x-i}{x+i}\right)$, and thence that $\pi=(2/i)\log(i)$.

71. Proposed by GUY B. COLLIER, 1901 Union, 27 Middle Section of South College, Schenectady, N. Y.

Find the locus of any point on the front sprocket of a bicycle during one revolution of the hind wheel (any gear may be assumed).

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

EDITORIALS.

Dr. Halsted reports that the Sylvester Fund has just been finished with about £900.

A copy each of the July number, Vol. II., and the July number, Vol. X., of the *Analyst* is wanted. Any one who can furnish these should write to the editor of the MONTHLY.

A number of important articles will appear in the MONTHLY during the next year. Dr. Miller will continue to furnish original matter on the all-embracing subject of Groups; Dr. E. O. Lovett will continue his series of articles on Lie's Transformation Groups; Dr. Roe has sent in a valuable article on Symmetric Functions, and Dr. Halsted has furnished a highly valuable account of the great Russian mathematician, Tchébychev. This biography will appear accompanied by an admirable picture of Tchébychev.

Nos. 6 and 11, of Vol. II. of the MONTHLY are exhausted. As we have yet a few orders for complete sets to fill, we shall be pleased to pay 40 cents apiece for copies of No. 6, Vol. II., and 30 cents apiece for copies of No. 11. There has been, of late, quite a demand for complete sets for College and University libraries. Whenever a sufficient number of orders are received to defray the expenses these numbers will be reprinted. Other numbers in other volumes of the MONTHLY are nearly exhausted. Subscribers desiring numbers to complete their files should order them at once.

In order to increase our subscription list for next year, we will allow each old subscriber who secures one new subscriber to remit us three dollars in full payment for one year's subscription to the MONTHLY for both.

BOOKS AND PERIODICALS.

Lectures on Elementary Mathematics. By Joseph Louis Lagrange. Being a course of lectures delivered at the École Normale, Paris, 1795. Translated from the French by Thomas J. McCormack. 8vo. Handsomely bound in Red Cloth. xvi+156 pages. Price, $1.00. Chicago: The Open Court Publishing Co.

In bringing out this valuable translation, the translator deserves great praise. These lectures which were delivered in 1795 and which are now found in the 7th volume of Lagrange's collected works have never before been published in separate form either in French or English, though a translation in German by Niedermüller appeared in 1880. These lectures are the source from which many of the best discussions of elementary mathematics have been drawn, besides containing much of great value and interest not yet incorporated in the text books. These lectures have received the indorsement of De Morgan, and Dühaing places them in the front rank of elementary expositions, as an exemplar of their kind.

The lectures discuss the theory of *Continued Fractions*, *Logarithms*, the *Operations of Arithmetic*, and fundamental principles generally. A brief *history of Algebra* is given and a full discussion of *equations of the third and fourth degree*, including the irreducible case. The two final lectures are devoted to the resolution of numerical equations, and to the usage of curves in the solution of problems.

Teachers of elementary mathematics, who desire to improve their methods of instruction, adding richness and vitality to the subject, should read this book. B. F. F.

A Manual of Experiments in Physics. Laboratory Instruction for College Classes. By Joseph S. Ames, Ph. D., Associate Professor of Physics in Johns Hopkins University, and William J. A. Bliss, Ph. D., Associate in Physics in Johns Hopkins University. 8vo. Cloth. 544 pages. New York: Harper & Brothers.

To my mind, this is by far the best laboratory manual that has appeared. B. F. F.

Elements of the Differential Calculus. By James McMahon, A. M. (Dublin), Assistant Professor of Mathematics in Cornell University, and Virgil Sny-

der, Ph. D. (Göttingen), Instructor in Mathematics in Cornell University. 8vo. Cloth. xiv+337 pages. Price, $2.00. New York, Cincinnati, and Chicago : American Book Co.

This book is the second of the Cornell Mathematical Series, of which series, Lucien Augustus Wait, Senior Professor of Mathematics in Cornell University, is editor.

The authors' apology for writing the book is that no other book has just the scope of this one, many of the works being too brief, and omitting rigorous proofs as being too difficult for the average student, while the more extensive treatises have too much for a student to master in the allotted time.

The authors have been very explicit in dealing with the fundamental principles of the science, and their efforts to reduce these principles to such clear and concise statements as to be readily comprehended by the student, will be greatly appreciated.

On page 59 we find the following method of differentiating $\log_a x$:

Let $y=\log_a x$. Then $y+\Delta y=\log_a(x+\Delta x)$.

$$\frac{\Delta y}{\Delta x}=\log_a\frac{(x+\Delta x)-\log_a x}{\Delta x}$$

$$=\frac{1}{\Delta x}\log_a\left(\frac{x+\Delta x}{x}\right)=\frac{1}{x}\cdot\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{x\,\div\,\Delta x}.$$

. . . . . .

$$\frac{dy}{dx}=\frac{1}{x}\lim_{\Delta x=0}\left[\log_a\left(1+\frac{\Delta x}{x}\right)\right]^{x/\Delta x}=\ldots\ldots\frac{1}{x}\log_a e.$$

It is then stated in a foot-note that this method is too brief to be rigorous, it assuming that $x/\Delta x$ is a positive integer, which is equivalent to restricting Δx to approach zero in a particular way. A rigorous and general proof is given in the Appendix, page 316, in which the above restriction is avoided. However, while the method of proof there given is rigorous, we believe the following to be equally so:

Let $y=\log_a x$, $a>1$. Then $y\pm\Delta y=\log_a(x\pm\Delta x)$,

the upper signs corresponding and the lower signs corresponding.

$$\therefore\ \pm\Delta y=\log_a(x\pm\Delta x)-\log_a x=\log_a\left(1\pm\frac{\Delta x}{x}\right).$$

$$\frac{\Delta y}{\Delta x}=\pm\frac{1}{\Delta x}\log_a\left(1\pm\frac{\Delta x}{x}\right)=\pm\frac{1}{\Delta x}\cdot\frac{x}{x}\log_a\left(1\pm\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1\pm\frac{\Delta x}{x}\right)^{\pm(x\,\div\,\Delta x)}.$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{x}\lim_{\Delta x=0}\log_a\left(1\pm\frac{\Delta x}{x}\right)^{\pm(x\,\div\,\Delta x)}=\frac{1}{x}\lim_{\Delta x=0}\log_a\left[\left(1^{\pm(x\,\div\,\Delta x)}+\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}\right)\right.\right.$$

$$\left.+\frac{1}{2!}\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}-1\right)\left(\pm\frac{\Delta x}{x}\right)^2+\frac{1}{3!}\left(\pm\frac{x}{\Delta x}\right)\left(\pm\frac{x}{\Delta x}-1\right)\left(\pm\frac{x}{\Delta x}-2\left(\pm\frac{\Delta x}{x}\right)^3\right.$$

$$\left.\left.+,\ \text{etc.}\right]=\frac{1}{x}\lim_{\Delta x=0}\log_a\left[1+1+\frac{1}{2!}\left(1\mp\frac{\Delta x}{x}\right)+\frac{1}{3!}\left(1\mp\frac{\Delta x}{x}\right)\left(1\mp\frac{\Delta x}{x}\right)+,\ \text{etc.}\right]\right.$$

$$=\frac{1}{x}\log_a\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+,\ \text{etc.}\right]=\frac{1}{x}\log_a e.$$

Δx is independent of x and can approach 0 in two ways only, viz., either from the left or from the right.

The same method holds when $a<1$, in which case $dy/dx=(1/x)\log_a(1/e)$.

The work is an excellent one and it together with the Integral Calculus of this series will make a most valuable course in this subject. We very heartily recommend this work to the consideration of teachers desiring something new and at the same time good.

B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. V. DECEMBER, 1898. No. 12.

BIOGRAPHY.

TCHÉBYCHEV.

BY GEORGE BRUCE HALSTED.

PAFNUTI LVOVICH CHEBYSHEV in the notice which appeared in THE AMERICAN MATHEMATICAL MONTHLY, Vol. II., pp. 61-63, shortly after his death, was ranked as second only to Lobachévski among Russian mathematicians. In the memoir of Vasiliev which has just appeared the author says that if Russian mathematics may claim an honorable place in the modern science it is due to Lobachévski, to Ostrogradski, to Chebyshev.

The biography of Sylvester in the Proceedings of the Royal Society says of Sylvester's discourse on Conversion of Motion: "A synopsis only of the Royal Institution lecture was published. The manuscript of the lecture as actually delivered is in the possession of George Bruce Halsted, of the University of Texas. Extracts from it appear in the American Journal 'Science', of April 16, 1897, from which it appears that it was characterized by that eloquence, force, and poetical imagination with which students of Sylvester are familiar." From this precious unpublished MS. of the great "king of thought" I give the following: "My friend, Professor Tchebicheff, of the University of St. Petersburg (one of the greatest mathematicians of this or any previous age, the only man ever able to cope with the refractory character and erratic flow of prime numbers and to confine the stream of their progression within algebraic limits, building up, if I

may so say, banks on either side which that stream, devious and irregular as are its windings, can never overflow), had long occupied himself with the theory of Parallel Motions. When he paid his first visit to me in England, at Lincoln's Inn Fields, about twenty years ago, his special object was to obtain sight of some of the old machines of Watt in order to ascertain if the parallel motions constructed by him followed strictly the description ordinarily given of them, or were adjusted in a manner to give more accurate results for a throw of the beam not indefinitely small, as the common theory virtually implies, but ranging within given limits.

I believe he found that the intuitive mechanical sagacity of Watt had anticipated his own mathematical deductions.

Since that time, among other researches of far greater importance, Tchebicheff has occupied himself with finding the best form of the 3-bar motion, of which Watt's parallel motion is only a special case, and arrived at results which give greater accuracy than Watt's at the same time that his arrangement is more compact, and, in some cases, more convenient of application. I may mention that in Tchebicheff's arrangement, the two radial bars cross instead of being parallel in the initial position; and that while in Watt's arrangement the parallel point (*i. e.* the part meant to move straight) describes a figure of 8, in Tchebicheff's system it describes a very flat but open oval curve.

As regards the problem of a perfect parallel motion, Tchebicheff had satisfied himself that the solution was impossible, and had elaborated what seemed to him *almost* a conclusive proof that such a machine was inconstructible in the nature of things,—almost but not quite; for under the very special hypothesis of two points usually distinct chancing to come into coincidence (if that could happen), he tells me that he foresaw that there was a bare possibility of his proof not holding water, and therefore with most praiseworthy caution, he held it back until he could succeed in patching up the supposed flaw. Still he made no secret of his inner conviction that the problem did not admit of solution; nor did he stand alone among mathematicians of the highest rank in entertaining that belief.

On his recent visit (the third I believe which he has paid) to England, in the course of last autumn, he again called upon me, and naturally, knowing how much the subject had occupied his thoughts, I inquired after his old love, and asked him how fared his proof of the impossibility of a Perfect Parallel Motion. 'Oh,' replied my eminent guest, 'it is done!' 'Done!' said I, offering him my hand, 'I may then congratulate you on having at last overcome the difficulty with which you have so long contended.' 'No,' said he, with a somewhat downcast air, 'it has been done in France and again in Russia.' 'Somebody else, then, has obtained a proof of the problem being impossible.' 'No; the problem has been solved, so that of course it cannot be demonstrated to be impossible.'

He then showed me Peaucellier's figure, which did indeed make me open my eyes, so simple—seeming to have so long eluded the gaze of the many sharp-sighted telescopic explorers watching to discover it flit over the wires! 'Now,